Partialbruchzerlegung

1.) $$\frac{1}{x^3-1} = \frac{1}{(x-1)(x^2+x+1)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+x+1}$$

$N(x) = x^3 - 1 = (x-1)(x^2+x+1)$

HORNER-Schema

Bestimmung der Koeffizienten A, B, C durch Koeffizienten-vergleich:

$$1 = A(x^2+x+1) + (Bx+C)(x-1)$$

d.h. $$1 = (A+B)x^2 + (A-B+C)x + A - C$$

x^2: $0 = A + B \rightsquigarrow B = -A$

x^1: $0 = A - B + C \qquad \rightsquigarrow 0 = A + A + A - 1 \rightsquigarrow A = \frac{1}{3}$

x^0: $1 = A - C \rightsquigarrow C = A - 1 \qquad B = -\frac{1}{3}$

$C = -\frac{2}{3}$

$$\frac{1}{x^3-1} = \frac{1/3}{x-1} - \frac{x+2}{3(x^2+x+1)} .$$

2.) $$\frac{x^3+1}{x(x-1)^3} = \frac{A}{x} + \frac{B_1}{x-1} + \frac{B_2}{(x-1)^2} + \frac{B_3}{(x-1)^3}$$ (Mehrfachwurzeln im Nenner)

Koeffizientenvergleich:

$$x^3+1 = A(x-1)^3 + B_1 x(x-1)^2 + B_2 x(x-1) + B_3 x$$

x^3: $1 = A + B_1 \qquad B_1 = 2$

x^2: $0 = -3A - 2B_1 + B_2 \qquad B_2 = 1$

x^1: $0 = 3A + B_1 - B_2 + B_3 \qquad B_3 = 2$

x^0: $1 = -A \qquad A = -1$

$$\frac{x^3+1}{x(x-1)^3} = -\frac{1}{x} + \frac{2}{x-1} + \frac{1}{(x-1)^2} + \frac{2}{(x-1)^3} .$$

I. (bekannt)

$$\int \frac{A}{x-a}\,dx = A\int \frac{dx}{x-a} = \underline{\underline{A\cdot \ln|x-a| + C}}$$

II. (bekannt)

$$\underset{(m>1)}{\int \frac{A}{(x-a)^m}\,dx} = A\int \frac{dx}{(x-a)^m} = \underline{\underline{A\cdot \frac{1}{(1-m)(x-a)^{m-1}} + C}}$$

III.

$$J(x) = \int \frac{Ax+B}{x^2+px+q} = \frac{A}{2}\int \frac{2x+p+\overbrace{\left(\frac{2B}{A}-p\right)}^{A_1}}{x^2+px+q}\,dx =$$

$(A \neq 0,\ \frac{p^2}{4}-q<0)$

$$= \overbrace{\frac{A}{2}\int \frac{(2x+p)\,dx}{\underbrace{x^2+px+q}}}^{J_1(x)} + \overbrace{\frac{A A_1}{2}\int \frac{dx}{x^2+px+q}}^{J_2(x)}$$

Zähler ist die Ableitung des Nenners

logarithm. Abl. $\rightsquigarrow \int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)| + C \rightsquigarrow J_1(x) = \frac{A}{2}\ln|x^2+px+q| + C$

$$J_2(x):\ x^2+px+q = \underbrace{\left(x+\frac{p}{2}\right)}_{\substack{u \\ du=dx}}{}^2 + \underbrace{\left(q-\frac{p^2}{4}\right)}_{=a^2\ (>0)} \rightsquigarrow J_2(x) = \frac{AA_1}{2}\int \frac{du}{u^2+a^2} =$$

denn $\frac{p^2}{4}-q<0$

11. Grundintegral

$$= \frac{AA_1}{2}\,\frac{1}{a}\arctan\frac{u}{a} + C$$

$$\rightsquigarrow \underline{\underline{J(x) = J_1(x) + J_2(x) = \frac{A}{2}\ln|x^2+px+q| + \frac{A\left(\frac{2B}{A}-p\right)}{2\sqrt{q-\frac{p^2}{4}}}\arctan\frac{x+\frac{p}{2}}{\sqrt{q-\frac{p^2}{4}}} + C}}$$