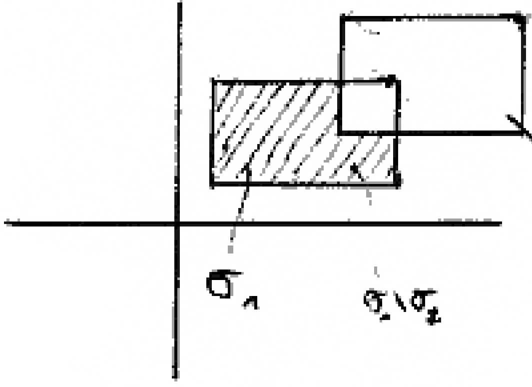
1. a)



Es gilt $(\sigma_1\setminus\sigma_2)\cap(\sigma_1\cap\sigma_2)=\emptyset$

und $\sigma_1=(\sigma_1\setminus\sigma_2)\cup(\sigma_1\cap\sigma_2)$

$$m(\sigma_1)=m[(\sigma_1\setminus\sigma_2)\cup(\sigma_1\cap\sigma_2)]=m(\sigma_1\setminus\sigma_2)+m(\sigma_1\cap\sigma_2)$$

$$\Rightarrow\ m(\sigma_1\setminus\sigma_2)=m(\sigma_1)-m(\sigma_1\cap\sigma_2)$$

Nun ist $\overline{\sigma_1\setminus\sigma_2}\supseteq\sigma_1\setminus\sigma_2$ also $m(\overline{\sigma_1\setminus\sigma_2})\geqq m(\sigma_1\setminus\sigma_2)$ (Meßbarkeit einer Menge: Borel-meßbar + L-Nullmenge des Randes)

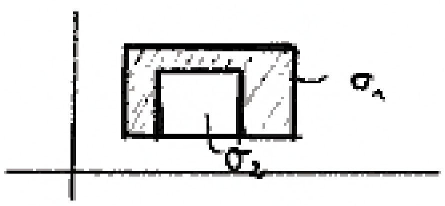
sowie $\sigma_1\cap\sigma_2\subseteq\sigma_2$ also $m(\sigma_1\cap\sigma_2)\leqq m(\sigma_2)$

also: $\underline{m(\overline{\sigma_1\setminus\sigma_2})}\geqq m(\sigma_1\setminus\sigma_2)=m(\sigma_1)-m(\sigma_1\cap\sigma_2)\geqq\underline{\underline{m(\sigma_1)-m(\sigma_2)}}$

ist $\sigma_2\subset\sigma_1$, dann gilt

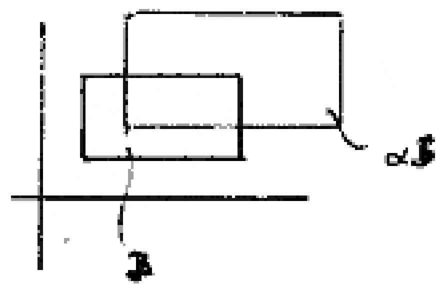
$\sigma_1=(\sigma_1\setminus\sigma_2)\uplus(\sigma_1\cap\sigma_2)=\sigma_1\setminus\sigma_2\cup\sigma_2$

also $m(\sigma_1)=m(\sigma_1\setminus\sigma_2)+m(\sigma_2)$, d.h. $m(\sigma_1\setminus\sigma_2)=m(\sigma_1)-m(\sigma_2)$



b) B JORDAN-mb. Menge

$\alpha>0 \qquad \alpha B=\{\alpha x : x\in B\}$



$x\in\mathbb{R}^n$

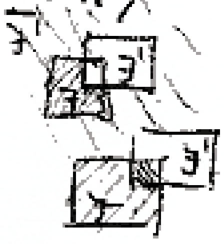
$\mathfrak{J}=\{(x_1\dots x_n)\in\mathbb{R}^n : a_k\leq x_k<b_k,\ k=1\dots n\}$

$|\mathfrak{J}|=m(\mathfrak{J})=\prod_{k=1}^{n}(b_k-a_k)$ bei endlichem $\mathfrak{J}$, sonst ∞.

Alle $\mathfrak{J}$ aus $\mathbb{R}^n$ bilden ein System S (also $\mathfrak{J}\in S$).

Diese bilden eine Algebra (d.h. die von S erzeugte Algebra) $\mathfrak{A}$,

also S erzeugt $\mathfrak{A}$.



$\mathfrak{A}=\{\emptyset, S, \bigcup_{k=1}^{l}\mathfrak{J}_k, \overline{\mathfrak{J}}\}$, wegen $\mathfrak{J}\setminus\mathfrak{J}'=\mathfrak{J}\cap\overline{\mathfrak{J}'}=\overline{\overline{\mathfrak{J}}\cup\mathfrak{J}'}\in\mathfrak{A}$

, wegen $\mathfrak{J}\cap\mathfrak{J}'=\mathfrak{J}\setminus(\mathfrak{J}\setminus\mathfrak{J}')\in\mathfrak{A}$

Algebra ist somit nie leer. Komplementbildung

αB : $\alpha x=\alpha(x_1\dots x_n)=(\alpha x_1\dots\alpha x_n)$

$$m(\alpha\mathfrak{J})=\prod_{k=1}^{n}(\alpha b_k-\alpha a_k)=\prod_{k=1}^{n}\alpha(b_k-a_k)=\prod_{k=1}^{n}\alpha\prod_{k=1}^{n}(b_k-a_k)=\alpha^n\prod_{k=1}^{n}(b_k-a_k)=\alpha^n m(\mathfrak{J})$$

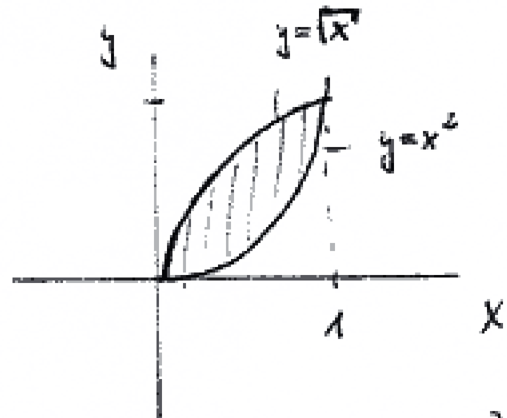
B JORDAN-mb heißt $B=\bigcup_{k=1}^{m}\mathfrak{J}_k$, $\mathfrak{J}_k\cap\mathfrak{J}_{k'}=\emptyset$, $\forall k\neq k'$.

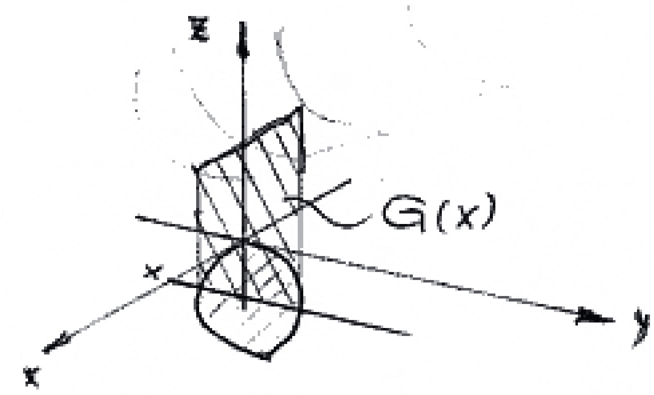
$m(B)=\sum_{k=1}^{m}m(\mathfrak{J}_k)$, da $m_i(B)=\sum m_i(\mathfrak{J}_k)=\sum m_e(\mathfrak{J}_k)=m_e(B)$ gilt

$m(\alpha B)=\sum_{k=1}^{m}m(\alpha\mathfrak{J}_k)=\sum_{k=1}^{m}\alpha^n m(\mathfrak{J}_k)=\alpha^n\sum_{k=1}^{m}m(\mathfrak{J}_k)=\alpha^n m(B)$

2. a) $\Omega \subset \mathbb{R}^2$: $y = x^2$, $y = \sqrt{x}$, $x \in [0,1]$

$K \subset \mathbb{R}^3$





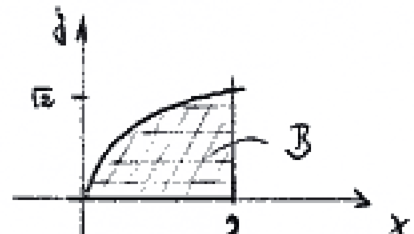
$K = \{(x,y,z) \in \mathbb{R}^3 : (x,y) \in \Omega,\ 0 \le z \le x^2 + y + 1\}$

$$G(x) = \int_{x^2}^{\sqrt{x}} (x^2 + y + 1)\,dy = x^2 y + \frac{1}{2}y^2 + y \Big|_{x^2}^{\sqrt{x}} = x^2\sqrt{x} + \frac{1}{2}x + \sqrt{x} - x^4 - \frac{1}{2}x^4 - x^2 = -\frac{3}{2}x^4 + x^2\sqrt{x} - x^2 + \frac{1}{2}x + \sqrt{x}$$

für $0 \le x \le 1$ $\qquad G(0) = 0$

$G(1) = -\frac{3}{2} + 1 - 1 + \frac{1}{2} + 1 = 0$

b) $B \subset \mathbb{R}^2$: $y = \sqrt{x}$, $y = 0$, $x = 2$

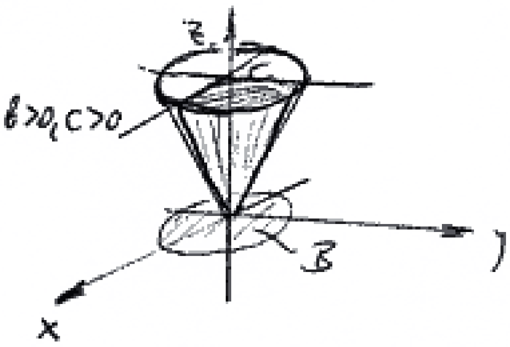


$$\int_{y=0}^{y=\sqrt{2}} \int_{x=y^2}^{x=2} x y^3\,dx\,dy = \int_0^{\sqrt{2}} \frac{1}{2}x^2 y^3 \Big|_{y^2}^{2} dy = \frac{1}{2}\int_0^{\sqrt{2}} (4y^3 - y^7)\,dy = \frac{1}{2}\left(y^4 - \frac{1}{8}y^8\right)\Big|_0^{\sqrt{2}} = \frac{1}{2}\left(4 - \frac{1}{8}\cdot 16\right) = 1$$

$$\int_{x=0}^{x=2} \int_{y=0}^{y=\sqrt{x}} x y^3\,dy\,dx = \int_0^2 \frac{1}{4}x y^4 \Big|_0^{\sqrt{x}} dx = \frac{1}{4}\int_0^2 x^3\,dx = \frac{1}{4}\ \frac{1}{4}x^4\Big|_0^2 = \frac{1}{16}\cdot 16 = 1$$

$$\iint_B x y^3\,dx\,dy = 1$$

c) A: $\left(\frac{z}{c}\right)^2 = \left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2$, $z = c$, $x = 0$, $y = 0$, $a>0, b>0, c>0$



$$J = \iint_B f(x,y)\,dx\,dy$$

$x = ra\cos t$

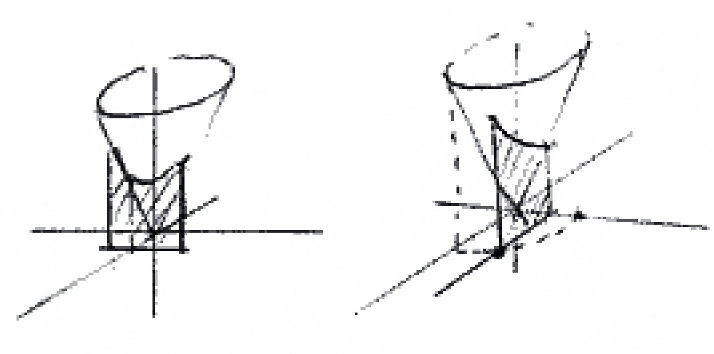
$y = rb\sin t$

$z = c\sqrt{\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2}$

$1 = r^2$

$$J = \int_0^{\pi/2} \int_0^1 c\,r\,ab\,r\,dr\,d\varphi$$

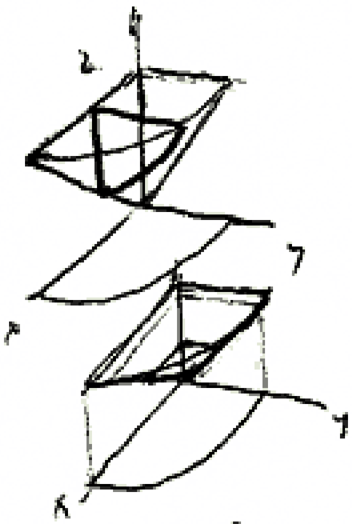
$$= abc\,\frac{\pi}{2}\,\frac{1}{3}r^3\Big|_0^1 = \underline{\frac{\pi abc}{6}}$$

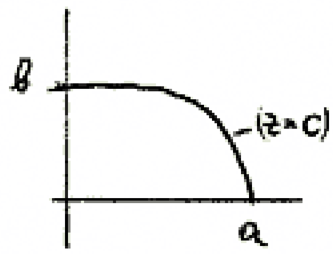


2c)

Integrationsgrenzen:

$$\frac{z^2}{c^2} = \frac{x^2}{a^2} + \frac{y^2}{b^2}$$





$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$$\left[\begin{array}{l} 0 \leq x \leq a \\ 0 \leq y \leq b\sqrt{1-\frac{x^2}{a^2}} \\ 0 \leq z \leq c\sqrt{\frac{x^2}{a^2}+\frac{y^2}{b^2}} \end{array}\right. \qquad \left\{\begin{array}{l} 0 \leq z \leq c \\ 0 \leq y \leq \frac{b}{c} z \\ 0 \leq x \leq a\sqrt{\frac{z^2}{c^2}-\frac{y^2}{b^2}} \end{array}\right.$$

2d)

$$\int\limits_{x=0}^{a} \int\limits_{y=0}^{b\sqrt{1-\frac{x^2}{a^2}}} \int\limits_{z=0}^{c\sqrt{\frac{x^2}{a^2}+\frac{y^2}{b^2}}} \frac{xy}{\sqrt{z}} \, dz\, dy\, dx = \int\limits_0^a \int\limits_0^{b\sqrt{1-\frac{x^2}{a^2}}} 2xy\sqrt{z}\,\Big|_0^{c\sqrt{\frac{x^2}{a^2}+\frac{y^2}{b^2}}} dy\, dx$$

$$\int\limits_{z=0}^{c} \int\limits_{y=0}^{\frac{b}{c}z} \int\limits_{x=0}^{a\sqrt{\frac{z^2}{c^2}-\frac{y^2}{b^2}}} \frac{xy}{\sqrt{z}} \, dx\, dy\, dz = \frac{1}{2}\int\limits_0^c \int\limits_0^{\frac{b}{c}z} x^2 \Big|_0^{a\sqrt{\frac{z^2}{c^2}-\frac{y^2}{b^2}}} \frac{y}{\sqrt{z}}\, dy\, dz = \frac{1}{2}\int\limits_0^c \int\limits_0^{\frac{bz}{c}} a^2\left(\frac{z^2}{c^2}-\frac{y^2}{b^2}\right)\frac{y}{\sqrt{z}}\, dy\, dz =$$

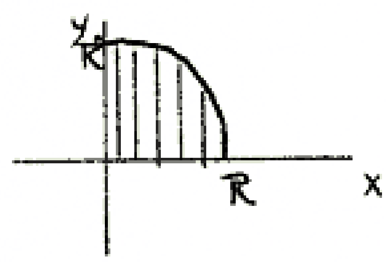
$$\frac{a^2}{2}\int\limits_0^c \left(\frac{1}{c^2} z^{3/2} \frac{1}{2} y^2 - \frac{1}{b^2} z^{-\frac{1}{2}} \frac{1}{4} y^4 \Big|_0^{\frac{bz}{c}}\right) dz = \frac{a^2}{2}\int\limits_0^c \left(\frac{1}{2c^2}\frac{b^2}{c^2} z^{\frac{7}{2}} - \frac{1}{4b^2}\frac{b^4}{c^4} z^{\frac{7}{2}}\right) dz = \frac{a^2}{2}\left(\frac{b^2}{2c^4}\frac{2}{9} z^{\frac{9}{2}} - \frac{b^2}{4c^4}\frac{2}{9} z^{\frac{9}{2}}\right)\Big|_0^c$$

$$\frac{a^2}{2}\left(\frac{b^2}{9c^4} c^{\frac{9}{2}} - \frac{b^2}{18c^4} c^{\frac{9}{2}}\right) = \frac{a^2}{2}\left(\frac{b^2}{9}\sqrt{c} - \frac{b^2}{18}\sqrt{c}\right) = \underline{\underline{\frac{1}{36} a^2 b^2 \sqrt{c}}}$$

3. $\iint\limits_{\Omega} x^2 y \, dx\, dy \qquad \Omega: x=0,\ y=0,\ x^2+y^2 = R^2,\ x \geq 0,\ y \geq 0$

a)

$$\int\limits_{x=0}^{R} \int\limits_{y=0}^{\sqrt{R^2-x^2}} x^2 y \, dy\, dx = \int\limits_0^R \frac{1}{2}x^2 y^2 \Big|_0^{\sqrt{R^2-x^2}} dx =$$



$$0 \leq x \leq R$$
$$0 \leq y \leq \sqrt{R^2-x^2}$$

$$\frac{1}{2}\int\limits_0^R x^2(R^2-x^2)\,dx = \frac{1}{2}\left(R^2\frac{1}{3}x^3 - \frac{1}{5}x^5\right)_0^R = \frac{1}{2}\left(\frac{R^5}{3} - \frac{R^5}{5}\right) = \underline{\underline{\frac{R^5}{15}}}$$

b)

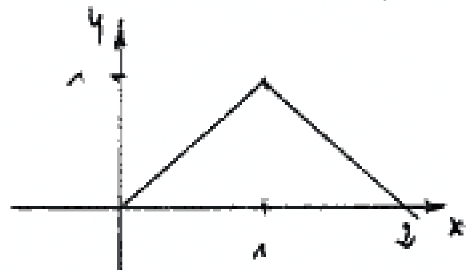
$$\int\limits_{y=0}^{R} \int\limits_{x=0}^{\sqrt{R^2-y^2}} x^2 y \, dx\, dy = \int\limits_0^R \frac{1}{3}x^3 \Big|_0^{\sqrt{R^2-y^2}} y\, dy = \frac{1}{3}\int\limits_0^R y\sqrt{R^2-y^2}^{\,3}\, dy =$$

$$0 \leq x \leq \sqrt{R^2-y^2}$$
$$0 \leq y \leq R$$

$$R^2 - y^2 = u$$
$$-2y\,dy = du$$

$$-\frac{1}{6}\int\limits_{R^2}^{0} \sqrt{u}^{\,3}\, du = -\frac{1}{6}\frac{2}{5} u^{\frac{5}{2}}\Big|_{R^2}^{0} = \frac{1}{15}\sqrt{R^2}^{\,5} = \underline{\underline{\frac{R^5}{15}}}$$

4. $\mathcal{J} = \iint\limits_{\Omega} (x+y)\,dx\,dy \qquad \Omega:\ y=x,\ y=2-x,\ y=0$

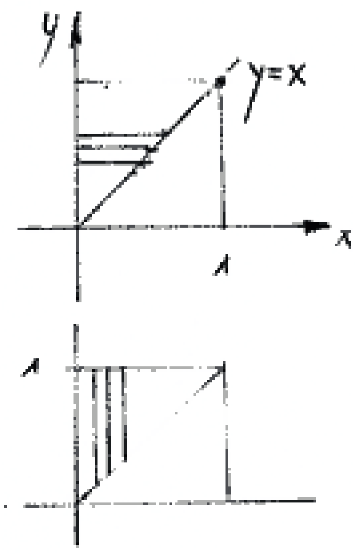


$y \le x \le 2-y$, $0 \le y \le 1$ | $0 \le x \le 1 \wedge 1 \le x \le 2$, $0 \le y \le x \wedge 0 \le y \le 2-x$

$$\int_0^1\int_y^{2-y} (x+y)\,dx\,dy = \int_0^1 \left(\frac12 x^2 + yx\Big|_y^{2-y}\right)dy = \frac12\int_0^1 [\underbrace{(2-y)^2-y^2}_{4-4y}]\,dy + \int_0^1 \underbrace{y[(2-y)-y]}_{2y-2y^2}\,dy =$$

$$2y - y^2\Big|_0^1 + y^2 - \frac23 y^3\Big|_0^1 = 2-1+1-\frac23 = \underline{\underline{\frac43}}$$

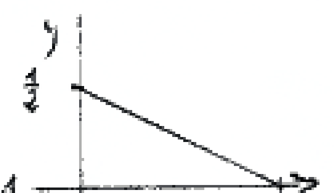
5. $\iint\limits_B e^{xy^2}\,dx\,dy \qquad B:\ 0 \le x \le 1,\ 0 \le x \le y$



$$\underbrace{\int_{y=0}^{1}\int_{x=0}^{x=y} e^{yx^2}\,dx\,dy}\ = \int_{x=0}^{1}\int_{y=x}^{1} e^{yx^2}\,dy\,dx = \int_0^1 (1-x)e^{y^2}\,dx$$

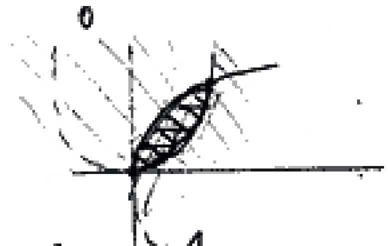
$$\int_0^1 e^{y^2} y\,dy \underset{\substack{y^2=u\\ 2y\,dy=du}}{=} \frac12\int_0^1 e^u\,du = \frac{e-1}{2}$$

6. a) $x + 2y \le 1,\ x \ge 0,\ y \ge 0$



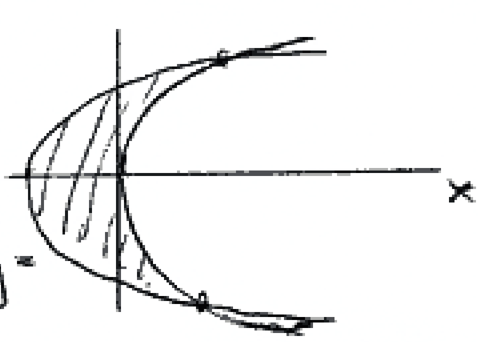
$$\int_0^{\frac12}\int_0^{1-2y} xy\,dx\,dy = \int_0^{\frac12} \frac12 x^2\Big|_0^{1-2y} y\,dy = \frac12\int_0^{\frac12}(1-2y)y\,dy = \frac12\left(\frac12 y^2 - \frac23 y^3\Big|_0^{\frac12}\right) = \frac12\left(\frac18 - \frac1{12}\right) = \frac18\left(\frac12 - \frac13\right) = \frac1{48}$$

b) $y \ge x^2 \wedge x \ge y^2$



$$\int_{x=0}^{1}\int_{y=x^2}^{\sqrt{x}} dy\,dx = \int_0^1 (\sqrt{x} - x^2)\,dx = \frac23\sqrt{x}^3 - \frac13 x^3\Big|_0^1 = \frac23 - \frac13 = \frac13$$

c) $3y^2 - 2 \le x \le y^2$



$3y^2 - 2 = y^2$
$2y^2 - 2 = 0$
$y^2 = 1 \qquad y = 1, -1$
$27y^6 - 39y^4 \cdot 2 + 3\cdot 3\cdot y^2\cdot 4 - 8$

$$\int_{-1}^{1}\int_{3y^2-2}^{y^2} x^2y^2\,dx\,dy = \int_{-1}^{1} \frac13 x^3 y^2\Big|_{3y^2-2}^{y^2} dy =$$

$$\frac13\int_{-1}^{1}[y^6 - (3y^2-2)^3]y^2\,dy = \frac13\int_{-1}^{1}(y^8 - 27y^8 + 54y^6 - 36y^4 + 8y^2)\,dy = \frac23\left[\frac19 y^9 - \frac{27}{9}y^9 + \frac{54}{7}y^7 - \frac{36}{5}y^5 + \frac83 y^3\right]_0^1 =$$

$$\frac23\left(-\frac{26}{9} + \frac{54}{7} - \frac{36}{5} + \frac83\right) = \frac{92}{315}$$

6. Vorbereitung: Zur Gammafunktion $\Gamma(x)=\int\limits_0^\infty e^{-t}\,t^{x-1}\,dt\,,\ x>0$

Es gilt 1. $\Gamma(x+1)=x\,\Gamma(x)$ und $\Gamma(n)=(n-1)!$

2. Ergänzungssatz $\Gamma(x)\,\Gamma(1-x)=\dfrac{\pi}{\sin\pi x}$

3. Verdoppelungssatz $\Gamma(x)\,\Gamma(x+\tfrac{1}{2})=\dfrac{\sqrt{\pi}}{2^{2x-1}}\,\Gamma(2x)$

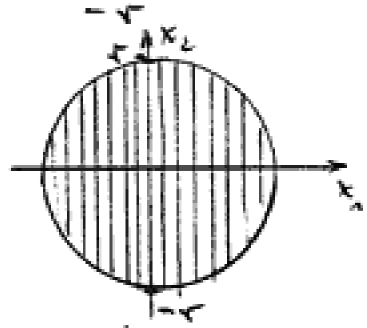
Desweiteren (Folgerungen aus 1–3): $\Gamma(\tfrac{1}{2})=\sqrt{\pi}$ $\qquad \Gamma(\tfrac{3}{2})=(\tfrac{1}{2})!=\tfrac{1}{2}\sqrt{\pi}$

Volumen $V_n(r)$:

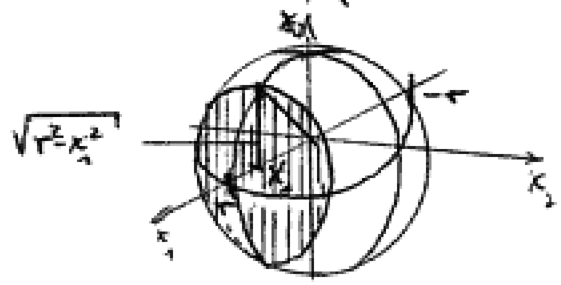
$n=1$	$V_1(r)=2r$	$V_1(1)=2$	
$n=2$	$V_2(r)=\pi r^2$	$V_2(1)=\pi$	$\Longrightarrow\ \underline{V_n(r)=V_n(1)\,r^n}$
$n=3$	$V_3(r)=\frac{4}{3}\pi r^3$	$V_3(1)=\frac{4}{3}\pi$	

$$V_n(r)=V_n(1)\,r^n=\int\limits_{x_1^2+\dots+x_n^2\le r^2}\!\!\cdots\int dx_1\cdots dx_n=\int\limits_{-r}^{r}\Bigg[\int\limits_{\sum_{\nu=2}^{n}x_\nu^2\le r^2-x_1^2}\!\!\cdots\int dx_2\cdots dx_n\Bigg]dx_1=\int\limits_{-r}^{r}V_{n-1}(1)\,(r^2-x_1^2)^{\frac{n-1}{2}}\,dx_1 \qquad (*)$$

$n=2$:
$$\iint\limits_{x_1^2+x_2^2\le r^2}dx_1\,dx_2=\int\limits_{-r}^{r}\Bigg[\int\limits_{x_2^2\le r^2-x_1^2}dx_2\Bigg]dx_1=\int\limits_{-r}^{r}\Bigg[\int\limits_{x_2=-\sqrt{r^2-x_1^2}}^{x_2=\sqrt{r^2-x_1^2}}dx_2\Bigg]dx_1$$



$n=3$:
$$\iiint\limits_{x_1^2+x_2^2+x_3^2\le r^2}dx_1\,dx_2\,dx_3=\int\limits_{-r}^{r}\Bigg[\iint\limits_{x_2^2+x_3^2\le r^2-x_1^2}dx_2\,dx_3\Bigg]dx_1$$



Weiterbehandlung von (*):

$$V_n(1)\,r^n=V_{n-1}(1)\int\limits_{-r}^{r}(r^2-x_1^2)^{\frac{n-1}{2}}\,dx_1\underset{\substack{x_1=rx\\dx_1=r\,dx}}{=}V_{n-1}(1)\,r^n\int\limits_{-1}^{1}(1-x^2)^{\frac{n-1}{2}}\,dx\underset{\substack{x=\sin\varphi\\dx=\cos\varphi\,d\varphi}}{=}V_{n-1}(1)\,r^n\int\limits_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos^n\varphi\,d\varphi \qquad (**)$$

Aus (**) folgt
$$\underline{\frac{V_n(1)}{V_{n-1}(1)}=\int\limits_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos^n\varphi\,d\varphi=J_n} \qquad ({*}{*}{*})$$

partielle Integration von (***): $J_n=(n-1)\int\limits_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos^{n-2}\varphi\,d\varphi-(n-1)\int\limits_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos^n\varphi\,d\varphi=(n-1)J_{n-2}-(n-1)J_n$

d.h. $\underline{n\,J_n=(n-1)\,J_{n-2}}\,,\ n\ge 2 \qquad (°)$

Daher ist für $n \geqq 2k+1$, $k \geqq 1$

$$n\frac{V_n(1)}{V_{n-2}(1)} = n\frac{V_n(1)}{V_{n-1}(1)}\frac{V_{n-1}(1)}{V_{n-2}(1)} = n\,\mathcal{J}_n\mathcal{J}_{n-1} = (n-2)\,\mathcal{J}_{n-2}\mathcal{J}_{n-3} = (n-4)\,\mathcal{J}_{n-4}\mathcal{J}_{n-5} = \dots =$$

$$(n-2k)\,\mathcal{J}_{n-2k}\mathcal{J}_{n-2k-1}.$$

Hieraus folgt für $n = 2m$, $k = m-1$ wegen $(*_*^*)$

$$n\frac{V_n}{V_{n-2}} = 2\mathcal{J}_2\mathcal{J}_1 = 2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos^2\varphi\,d\varphi\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos\varphi\,d\varphi = 2\pi$$

also

$$\underline{V_n = \frac{2\pi}{n}V_{n-2}} \qquad (\overset{\circ}{\circ})$$

Dasselbe ergibt sich aus $(*_*^*)$ und $(^\circ)$ mit $n = 2m+1$, $k = m$.

Aus $(\overset{\circ}{\circ})$ folgt

$$V_n(1) = \frac{(2\pi)^k}{n(n-2)\cdot\ldots\cdot(n-2k+2)}V_{n-2k} = \pi^k\frac{2^k}{2^k\frac{n}{2}(\frac{n}{2}-1)\cdot\ldots\cdot(\frac{n}{2}-[k-1])}V_{n-2k}$$

$$= \pi^k\frac{(\frac{n}{2}-k)!}{\frac{n}{2}(\frac{n}{2}-1)\cdot\ldots\cdot(\frac{n}{2}-k+1)(\frac{n}{2}-k)!}V_{n-2k} = \pi^k\frac{\Gamma(\frac{n}{2}-k+1)}{\Gamma(\frac{n}{2}+1)}V_{n-2k} \qquad (\overset{\circ\,\circ}{\circ})$$

Aus $(\overset{\circ\,\circ}{\circ})$ folgt dann für $n = 2m$, $k = m-1$

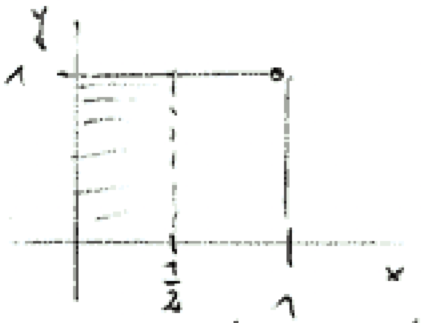
$$V_n(1) = \frac{\pi^{m-1}\Gamma(2)}{\Gamma(\frac{n}{2}+1)}V_2(1) = \frac{\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2}+1)}$$

und für $n = 2m+1$, $k = m$

$$V_n(1) = \frac{\pi^m\Gamma(\frac{3}{2})}{\Gamma(\frac{n}{2}+1)}V_1(1) = \frac{\pi^m\cdot\frac{1}{2}\sqrt{\pi}}{\Gamma(\frac{n}{2}+1)}\cdot 2 = \frac{\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2}+1)}$$

Ergebnis: $$\underline{V_n(r) = V_n(1)\,r^n = \frac{\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2}+1)}r^n}$$

7.



$$f(x,y)=\begin{cases} 1 & , 0\le x\le \frac12, y\in\mathbb{R}\setminus\mathbb{Q} \\ 0 & , 0\le x\le 1, y\in\mathbb{Q} \\ -1 & , \frac12 < x \le 1, y\in\mathbb{R}\setminus\mathbb{Q}\end{cases}$$

Schluß von einem iterierten Integral auf die Existenz des Mehrfachintegrals oder des anderen iterierten Integrals i.A. nicht statthaft.
(Schluß vom Mehrfachintegral auf Ex. der iterierten Integrale i.A. nicht statthaft)

zu: $\int_0^1\left(\int_0^1 f(x,y)\,dx\right)dy \overset{(*)}{=} 0$

$\int_0^1 f(x,y)\,dx = \qquad = 0 \quad (*)$

y irrat. $S^* = \sum_{i=1}^n \sup_{\substack{x\in[x_{i-1},x_i]\\ 0\le x\le \frac12}} f(x,y)(x_i-x_{i-1}) + \sum_{i=1}^n \sup_{\substack{x\in[x_{i-1},x_i]\\ \frac12 < x\le 1}} f(x,y)(x_i-x_{i-1}) = \sum_{i=1}^n 1\cdot(x_i-x_{i-1}) + \sum(-1)(x_i-x_{i-1}) = \frac12-\frac12 = 0$

$s_* = \sum_{0\le x\le\frac12} \inf f(x,y)(x_i-x_{i-1}) + \sum_{\frac12<x\le 1} \inf f(x,y)(x_i-x_{i-1}) = \sum 1\,(x_i-x_{i-1}) + \sum(-1)(x_i-x_{i-1}) = \qquad 0$

y rat. $S^* = \sum \sup f\,(x_i-x_{i-1}) = 0$
$s_* = 0$

zu: $\int_0^1\left(\int_0^1 f(x,y)\,dy\right)\cdot dx$ ex. nicht wegen $(+)$, $(**)$

$\int_0^1 f(x,y)\,dy$ für $x\in[0,\frac12]$

$0 = \underbrace{\sum_{i=1}^n \underbrace{\inf_{y\in[y_{i-1},y_i]} f(x,y)}_{=0}(y_i-y_{i-1})}_{=s_*(f)} \le \int_0^{1} f(x,y)\,dy \le \underbrace{\sum \underbrace{\sup_{y\in[y_{i-1},y_i]} f(x,y)}_{=1}(y_i-y_{i-1})}_{=S^*(f)} = 1$ ex. nicht. $(+)$

für $x\in(\frac12,1]$

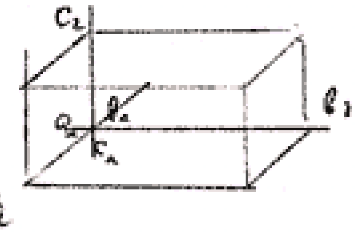
$-1 = \underbrace{\sum \underbrace{\inf f(x,y)}_{=-1}(y_i-y_{i-1})}_{=s_*(f)} \le \int_{0}^{1} f(x,y)\,dy \le \underbrace{\sum \underbrace{\sup f(x,y)}_{=0}(y_i-y_{i-1})}_{=S^*(f)} = 0$ ex. nicht $(**)$

zu: $J = \iint_B f(x,y)\,db$ Für $\varepsilon>0$ bel. gilt $|f(x,y)|>\varepsilon$ erfüllen überabzählbar viele (x,y), y irrat.
d.h. $S_*(f) = \sum_{i,j=1}^n \inf_{(x,y)\in\Delta b_{ij}} f(x,y)\,\Delta b_{ij} = -\sum \Delta b_{ij} = -B$, $S^*(f) = \sum \underbrace{\sup f}_{=1}\,\Delta b_{ij} = \sum \Delta b_{ij} = B$

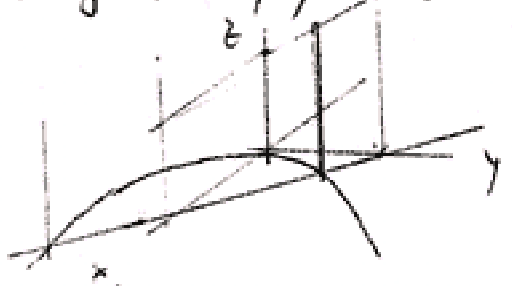
d.h. $S_* \le J \le S^{*} = -1$ Daraus folgt $-B \le J \le B$ für alle Zerlegungen $\Rightarrow J$ ex. nicht

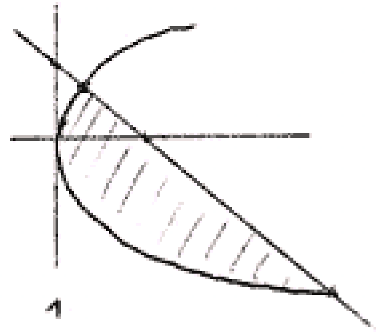
8. $\iiint\limits_{\Omega} f(x,y,z)\,dx\,dy\,dz = \int\limits_{X} f_1(x)\,dx \int\limits_{Y} f_2(y)\,dy \cdot \int\limits_{Z} f_3(z)\,dz$ když $f = f_1 \cdot f_2 \cdot f_3$

obecně $\int\limits_{c_1}^{c_2}\left[\int\limits_{b_1}^{b_2}\left\{\int\limits_{a_1}^{a_2} f(x,y,z)\,dx\right\}dy\right]dz$ odw to.



9. a) $x+y-2=0$, $y-z+2=0$, $y^2-x=0$, $z=0$





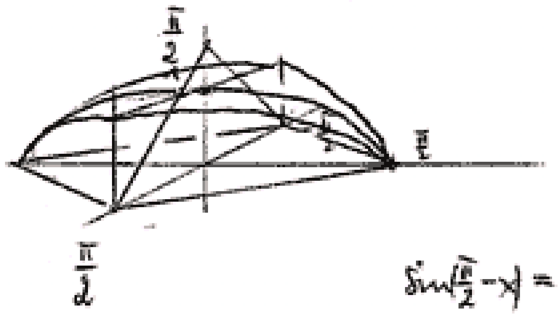
$y^2+y-2=0$

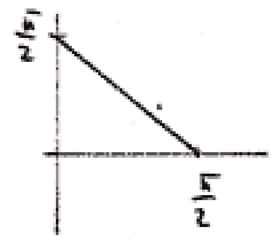
$y_{1/2} = -\frac{1}{2} \pm \sqrt{\frac{1}{4}+2} = -\frac{1}{2} \pm \frac{3}{2} = \begin{cases} 1 \\ -2 \end{cases}$

$\int\limits_{-2}^{1}\int\limits_{y^2}^{2-y}(y+2)\,dx\,dy = \int\limits_{-2}^{1} x(y+2)\Big|_{y^2}^{2-y}dy = \int\limits_{-2}^{1}(2-y-y^2)(y+2)\,dy = \int\limits_{-2}^{1}(2y+4-y^2-2y-y^3-2y^2)\,dy =$

$\int\limits_{-2}^{1}(4-3y^2-y^3)\,dy = 4y - y^3 - \frac{1}{4}y^4\Big|_{-2}^{1} = 12 - 9 + \frac{15}{4} = \underline{\frac{27}{4}}$

b) $|x|+|y| = \frac{\pi}{2}$, $z=0$, $z=\cos y$



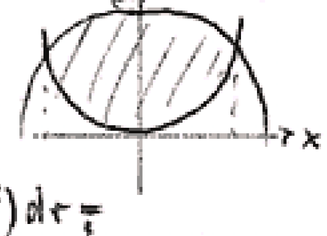


$V = 4\int\limits_{x=0}^{\frac{\pi}{2}}\int\limits_{y=0}^{\frac{\pi}{2}-x}\cos y\,dy\,dx = 4\int\limits_{0}^{\frac{\pi}{2}}\sin\left(\frac{\pi}{2}-x\right)dx$

$\sin\left(\frac{\pi}{2}-x\right) = \sin\frac{\pi}{2}\cos x - \cos\frac{\pi}{2}\sin x = \cos x$

$= 4\int\limits_{0}^{\frac{\pi}{2}}\cos x\,dx = 4\sin x\Big|_{0}^{\frac{\pi}{2}} = \underline{4}$

c) $x^2+y^2 = 2z$, $x^2+y^2+z^2 = 3$



$x = r\cos\varphi$ $\quad z = \frac{1}{2}r^2$

$y = r\sin\varphi$ $\quad z = \sqrt{3-r^2}$

$z = z$

$\frac{1}{4}r^4 = 3 - r^2$

$r^4 + 4r^2 - 12 = 0$

$u^2 + 4u - 12 = 0$

$u_{1/2} = -2 \pm 4 = \begin{cases} 2 \\ -6 \end{cases}$

$r^2 = 2 \quad r = \sqrt{2}$

$V = \int\limits_{0}^{2\pi}\int\limits_{0}^{\sqrt{2}}\int\limits_{\frac{1}{2}r^2}^{\sqrt{3-r^2}} r\,dz\,dr\,d\varphi = 2\pi\int\limits_{0}^{\sqrt{2}} r\left(\sqrt{3-r^2} - \frac{1}{2}r^2\right)dr =$

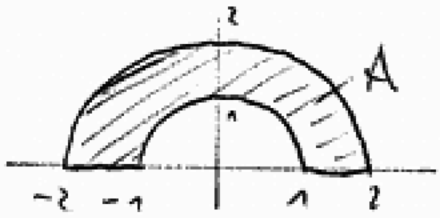
$3 - r^2 = t$
$-2r\,dr = dt$

$2\pi\left(-\frac{1}{2}\frac{2}{3}\sqrt{t}^3 - \frac{1}{2}\frac{1}{4}r^4\right)\Big|_{0}^{\sqrt{2}} =$

$2\pi\left(-\frac{1}{3}\sqrt{3-r^2}^3 - \frac{1}{8}r^4\right)\Big|_{0}^{\sqrt{2}} = -2\pi\left(\frac{1}{3} - \frac{\sqrt{3}^3}{3} + \frac{1}{2}\right) = -2\pi\,\frac{2-2\sqrt{3}^3+3}{6} =$

$-\frac{\pi}{3}\left(5-2\sqrt{3}^3\right) = \underline{\frac{\pi}{3}\left(6\sqrt{3}-5\right)}$

11.



$$J = \iint_A (x+y)\,dx\,dy$$

$x = r\cos\varphi \quad y = r\sin\varphi \qquad 1 \le r \le 2, \quad 0 \le \varphi \le \pi$

$$\int_0^{\pi}\int_1^2 r^2(\cos\varphi + \sin\varphi)\,dr\,d\varphi = \int_0^{\pi} \frac{1}{3} r^3 (\cos\varphi + \sin\varphi)\,d\varphi \Big|_1^2 = \frac{7}{3}\int_0^{\pi}(\cos\varphi + \sin\varphi)\,d\varphi =$$

$$\frac{7}{3}\left[\sin\varphi - \cos\varphi \Big|_0^{\pi}\right] = -\frac{7}{3}(-1-1) = \underline{\underline{\frac{14}{3}}}$$

12.

$$\varphi_1 = \iint_{A_2} \frac{\cos\beta_1 \cos\beta_2}{\pi s^2}\,dA_2$$

$x = r\cos\varphi \qquad 0 \le r \le b$

$y = r\sin\varphi \qquad 0 \le \varphi \le 2\pi$

$s^2 = a^2 + r^2$

$\cos\beta_1 = \frac{a}{s} \qquad \cos\beta_2 = \frac{a}{s}$

$$\varphi_1 = \int_0^{2\pi}\int_0^b \frac{a^2}{\pi s^4}\, r\,dr\,d\varphi = \frac{a^2}{\pi}\int_0^{2\pi}\int_0^b \frac{r\,dr}{(a^2+r^2)^2}\,d\varphi = 2a^2 \int_0^b \frac{r\,dr}{(a^2+r^2)^2} = a^2 \int_{a^2}^{a^2+b^2} \frac{du}{u^2} =$$

$a^2 + r^2 = u$

$2r\,dr = du$

$$-a^2 \frac{1}{\cancel{3}\, u^{\cancel{3}}}\Big|_{a^2}^{a^2+b^2} = -\frac{a^2}{\cancel{3}}\left[\frac{1}{(a^2+b^2)^{\cancel{3}}} - \frac{1}{a^{\cancel{6}}}\right] = \cancel{-\frac{a^2}{3}\,\frac{a^6 - (a^2+b^2)^3}{a^6(a^2+b^2)^3}} = \cancel{\frac{1}{3a^4}\,\frac{a^6 - a^6 - 3a^4b^2 - 3a^2b^4 - b^6}{(a^2+b^2)^3}} =$$

$$\cancel{+\frac{1}{3a^4}\,\frac{b^2(3a^4+3a^2b^2+b^4)}{(a^2+b^2)^3}} \quad -a^2\,\frac{a^2 - a^2 - b^2}{a^2(a^2+b^2)} = \underline{\underline{\frac{b^2}{a^2+b^2}}}$$

13.

$$x^2+y^2+z^2 \le 2az \Rightarrow x^2+y^2+z^2-2az \le 0 \Rightarrow x^2+y^2+(z-a)^2 \le a^2 \qquad M(0,0,a),\ r = a$$

$x = r\cos\varphi\sin\vartheta$

$y = r\sin\varphi\sin\vartheta \qquad dV = r^2\sin\vartheta\,dr\,d\vartheta\,d\varphi$

$z = r\cos\vartheta + a$

$$\begin{vmatrix} \cos\varphi\sin\vartheta & r\cos\vartheta\cos\varphi & -r\sin\vartheta\sin\varphi \\ \sin\varphi\sin\vartheta & r\cos\vartheta\sin\varphi & r\sin\vartheta\cos\varphi \\ \cos\vartheta & -r\sin\vartheta & 0 \end{vmatrix} =$$

$0 \le r \le a$

$0 \le \varphi \le 2\pi$

$0 \le \vartheta \le \pi$

$r^2\cos^2\vartheta\sin\vartheta\cos^2\varphi + r^2\sin^3\vartheta\sin^2\varphi + r^2\sin\vartheta\cos^2\vartheta\sin^2\varphi$

$r^2\sin^3\vartheta\cos^2\varphi = r^2\sin\vartheta$

$$M = \int_0^{2\pi}\int_0^{\pi}\int_0^a \sqrt{r^2\sin^2\vartheta + r^2\cos^2\vartheta + 2ar\cos\vartheta + a^2}\; r^2\sin\vartheta\,dr\,d\vartheta\,d\varphi = 2\pi\int_0^{\pi}\int_0^a \sqrt{r^2 + 2ar\cos\vartheta + a^2}\; r^2\sin\vartheta\,dr\,d\vartheta$$

$r^2 + 2ar\cos\vartheta + a^2 = u$

$-2ar\sin\vartheta\,d\vartheta = du$

$$2\pi\int_0^a\int_{(r+a)^2}^{(r-a)^2} \sqrt{u}\, r^2\left(-\frac{1}{2ra}\right)du\,dr = \frac{\pi}{a}\int_0^a\int_{(r-a)^2}^{(r+a)^2} r\sqrt{u}\,du\,dr = \frac{2\pi}{3a}\int_0^a r\sqrt{u}^3\Big|_{(r-a)^2}^{(r+a)^2} dr = \frac{2\pi}{3a}\int_0^a r\left(|r+a|^3 - |r-a|^3\right)dr =$$

$$|r+a|^3-|r-a|^3 = (r+a)^2+(r-a)^3 = 2r^3+6ra^2$$

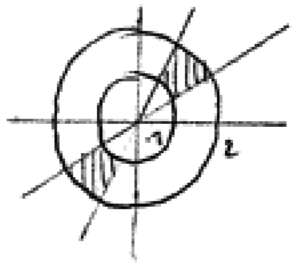
$! \; r-a<0$

$$\frac{2}{3}\frac{\pi}{a}\int_0^a (2r^3+6a^2r)\,r\,dr = \frac{4\pi}{3a}\left(\frac{r^5}{5}+\frac{3a^2r^3}{3}\right)_0^a = \frac{4\pi}{3a}\cdot\frac{6a^5}{5} = \underline{\underline{\frac{8\pi}{5}a^4}}$$

usw.

14. a) $y=x \qquad x^2+y^2=1$

$y=\sqrt{2}\,x \qquad x^2+y^2=2$



$x = r\cos\varphi$
$y = r\sin\varphi$

$$2\int_{\frac{\pi}{4}}^{\arctan\sqrt{2}}\int_1^2 r^2\cdot r\,dr\,d\varphi =$$

$y=x \Leftrightarrow \varphi=\frac{\pi}{4}$

$r\sin\varphi=\sqrt{2}\,r\cos\varphi \Leftrightarrow \tan\varphi=\sqrt{2}$

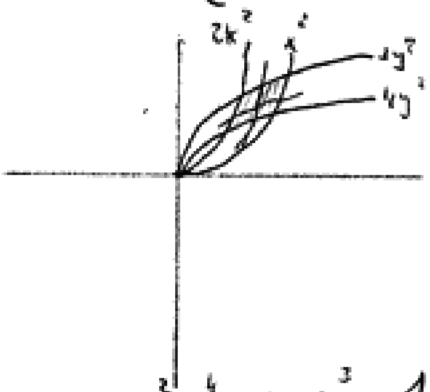
$$2\left(\arctan\sqrt{2}-\frac{\pi}{4}\right)\frac{r^4}{4}\Big|_1^2 = \frac{15}{2}\left(\arctan\sqrt{2}-\frac{\pi}{4}\right) = \frac{15}{2}(0{,}9553-0{,}7\ldots) = \frac{15}{2}\cdot 0{,}1699184 = \underline{\underline{1{,}274383}}$$

b) $y=x^2 \qquad x=3y^2$

$y=2x^2 \qquad x=4y^2$

$y=px^2 \qquad x=qy^2$

$1\le p\le 2\,, \quad 3\le q\le 4$



$y = pq^2y^4 \Rightarrow y^3=\frac{1}{pq^2} \qquad y=\frac{1}{\sqrt[3]{pq^2}}$

$x = qp^2x^4 \Rightarrow x^3=\frac{1}{qp^2} \qquad x=\frac{1}{\sqrt[3]{qp^2}}$

$p=\frac{y}{x^2} \qquad q=\frac{x}{y^2}$

$$\frac{\partial(p,q)}{\partial(x,y)} = \begin{vmatrix} -\frac{2y}{x^3} & \frac{1}{x^2} \\ \frac{1}{y^2} & -\frac{2x}{y^3}\end{vmatrix} = \frac{4xy}{x^3y^3}-\frac{1}{x^2y^2} = \frac{3}{x^2y^2} = \frac{3pq}{y\,x} = \frac{3pq}{\frac{1}{\sqrt[3]{qp^2}}\,\frac{1}{\sqrt[3]{pq^2}}} = 3pq\sqrt[3]{q^3p^3} = 3p^2q^2$$

$$\frac{\partial(x,y)}{\partial(p,q)} = \frac{1}{3p^2q^2}$$

$$\frac{1}{3}\int_3^4\int_1^2 \frac{1}{p^2q^2}\,dp\,dq = \frac{1}{3}\left(-\frac{1}{p}\right)_1^2\left(-\frac{1}{q}\right)_3^4 = \frac{1}{3}\left(\frac{1}{2}-1\right)\left(\frac{1}{4}-\frac{1}{3}\right) = \frac{1}{3}\,\frac{-1}{2}\,\frac{-1}{12} = \underline{\underline{\frac{1}{72}}}$$