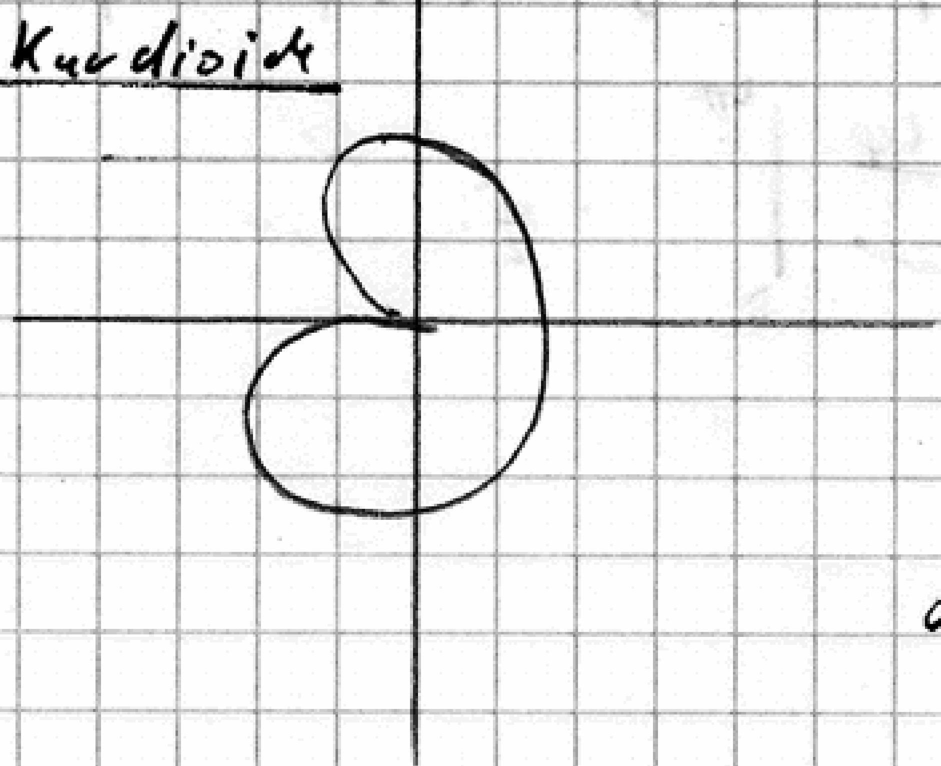
5. Übung

1) Kardioide



$$\rho(\varphi) = a(1+\cos\varphi)$$

$$\vec{r} = \rho(\varphi)\vec{e}_\rho$$

$$dr = \left(\frac{d\rho}{d\varphi}\vec{e}_\rho + \rho\frac{d\vec{e}_\rho}{d\varphi}\right)d\varphi$$

$$= \left(\frac{d\rho}{d\varphi}\vec{e}_\rho + \rho\vec{e}_\varphi\right)d\varphi$$

$$ds = |d\vec{r}| = \sqrt{\left|\frac{d\rho}{d\varphi}\right|^2 + \rho^2}\, d\varphi$$

$$s = \oint ds = \int_0^{2\pi} d\varphi \sqrt{a^2\sin^2\varphi + a^2(1+\cos\varphi)^2}$$

$$= \sqrt{2}\, a\, 2\int_0^{\pi} d\varphi \sqrt{1+\cos\varphi} \qquad \text{subst } u = \cos\varphi$$

$$s = 2\sqrt{2}\, a \int_{-1}^{1} du \frac{\sqrt{1+u}}{\sqrt{1-u^2}}$$

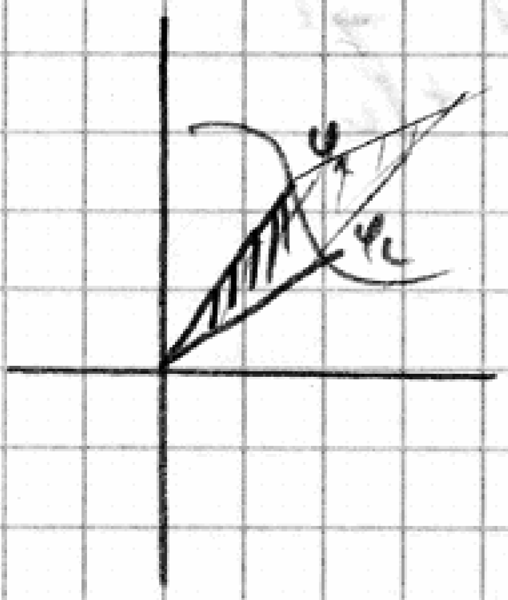
$$= 2\sqrt{2}\, a \int_{-1}^{1} \frac{du}{\sqrt{1-u}}$$

$$s = 2\sqrt{2}\, a \left(-2\sqrt{1-u}\Big|_{-1}^{+1}\right) = \underline{\underline{8a}}$$

oder: $\int_0^\pi d\varphi \sqrt{1+\cos\left(\frac{\varphi}{2}+\frac{\varphi}{2}\right)} = \int d\varphi \sqrt{\underbrace{1}_{\cos^2\frac{\varphi}{2}+\sin^2\frac{\varphi}{2}}+\cos^2\frac{\varphi}{2}-\sin^2\frac{\varphi}{2}}$

$$= \int d\varphi \sqrt{2\cos^2\frac{\varphi}{2}} = \sqrt{2}\int_0^\pi d\varphi \cos\frac{\varphi}{2}$$

b)



$$\frac{|r(\varphi_2) \times (r(\varphi_2) + r(\varphi_1))|}{2} = \frac{|r(\varphi_2) \times r(\varphi_1)|}{2}$$

$$\sum_i \frac{|r(\varphi_i) \times \Delta r_i|}{2}\frac{\Delta\varphi}{\Delta\varphi} \Rightarrow \frac{1}{2}\int\left|r \times \frac{dr}{d\varphi}\right| d\varphi$$

$$\left|r \times \frac{dr}{d\varphi}\right| = \left|\rho(\varphi) \times \left(\frac{d\rho}{d\varphi}\vec{e}_\rho + \rho\vec{e}_\varphi\right)\right|$$

$$= \left|\underbrace{\rho(\varphi)\vec{e}_\rho \times \frac{d\rho}{d\varphi}\vec{e}_\rho}_{0} + \rho^2 \underbrace{\vec{e}_\rho \times \vec{e}_\varphi}_{\vec{e}_z}\right| = \rho^2$$

$$A = \frac{1}{2}\int_0^{2\pi} \rho^2 \, d\varphi = \frac{1}{2}\int_0^{2\pi} a^2 (1+\cos\varphi)^2 \, d\varphi = \frac{a^2}{2}\int 1 + 2\cos\varphi + \cos^2\varphi \, d\varphi$$

$$= \varphi + 2\sin\varphi + \underbrace{\frac{\sin 2\varphi + 2\varphi}{4}}_{\int \cos^2\varphi} \Bigg|_0^{2\pi} \cdot \frac{a^2}{2}$$

$$= 3\pi \frac{a^2}{2}$$

2) a) 2 Teilchen m_1, m_2
$\vec{p}_1, \vec{p}_2$

$Q \quad m_1, \vec{p}_1\,' , m_2 \vec{p}_2\,'$

Labor:

$$\vec{p}_1 + \vec{p}_2 = \vec{p}_1\,' + \vec{p}_2\,'$$

~~CMS: Center mass system~~

$\vec{v}_s =$

$$\frac{\vec{p}_1^{\,2}}{2m_1} + \frac{\vec{p}_2^{\,2}}{2m_2} = \frac{\vec{p}_1^{\,\prime 2}}{2m_1} + \frac{\vec{p}_2^{\,\prime 2}}{2m_2} + Q$$

CMS: Center mass system (Schwerpunktsystem)

$$\vec{v}_s = \frac{m_1\vec{v}_1 + m_2\vec{v}_2}{m_1+m_2}$$

$$\vec{r}_1 = \vec{r}_s + \vec{R}_1 \qquad \dot{\vec{r}}_s = \vec{v}_s = \text{const}$$

$$\vec{r}_2 = \vec{r}_s + \vec{R}_2$$

$$\vec{v}_s = \frac{m_1\vec{v}_1 + m_2\vec{v}_2}{m_1+m_2} = \frac{\vec{p}_1 + \vec{p}_2}{m_1+m_2} = \frac{\vec{p}_1\,' + \vec{p}_2\,'}{m_1+m_2} = \vec{v}_s \quad \text{im Laborsystem}$$

im CMS $\to \vec{v}_s = 0$

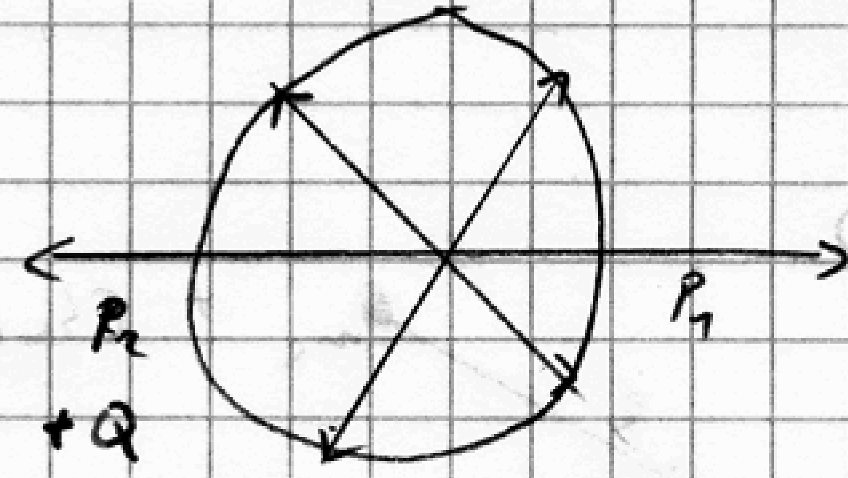
$$\dot{\vec{r}}_1 = \dot{\vec{r}}_s + \dot{\vec{R}}_1$$

$$m\dot{\vec{R}}_1 + m\dot{\vec{R}}_2 = 0 \qquad \vec{p}_1 + \vec{p}_2 = 0 = \vec{p}_1\,' + \vec{p}_2\,'$$

CMS: $$\frac{p_1^2}{2m_1} + \frac{p_2^2}{2m_2} = \frac{p_1'^2}{2m_1} + \frac{p_2'^2}{2m_2}$$

b)

CMS: $\vec{p}_2 = -\vec{p}_1 \qquad \vec{p}_2\,' = -\vec{p}_1\,'$



$$p_1^2\left(\frac{1}{2m_1}+\frac{1}{2m_2}\right) = p_1'^2\underbrace{\left(\frac{1}{2m_1}+\frac{1}{2m_2}\right)}_{\frac{1}{2\mu}} + Q$$

$$|p_1'| = \sqrt{p_1^2 - 2\mu Q} \qquad p_1' = |p_1'|\,\hat{e}$$

Labor: $\vec{P}_2 = 0 \qquad \vec{P}_1 = \vec{p} \qquad V_S = \frac{\vec{P}_1}{m_1+m_2}$

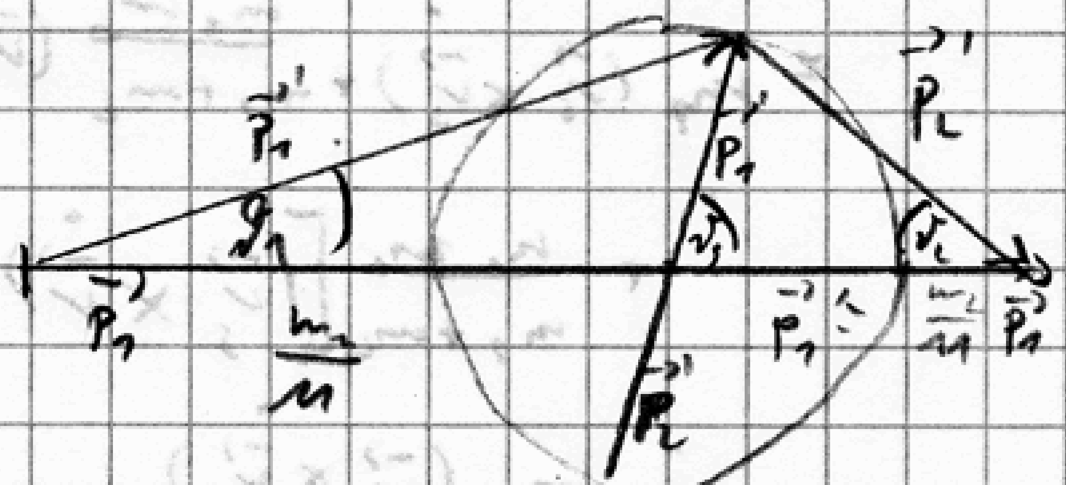
$$\vec{P}_1\,' = m_1\vec{V}_1\,' = m_1(\vec{V}_S + \vec{v}_1\,') = \frac{m_1}{M}\vec{P}_1 + |\vec{p}_1\,'|\,\hat{e}$$

$$\vec{P}_2\,' = m_2\vec{V}_2\,' = m_2(\vec{V}_S + \vec{v}_2\,') = \frac{m_2}{M}P_1 - |\vec{p}_1\,'|\,\hat{e}$$

$$\vec{p}_1 = m_1\vec{v}_1 = m_1(\vec{V}_1 - \vec{V}_S) = P_1 - m_1 V_S$$

$$\cancel{\vec{p}_2\cdot \quad = P_1 - \frac{m_1}{M}P_1 = \frac{m_2}{M}\vec{P}_1}$$

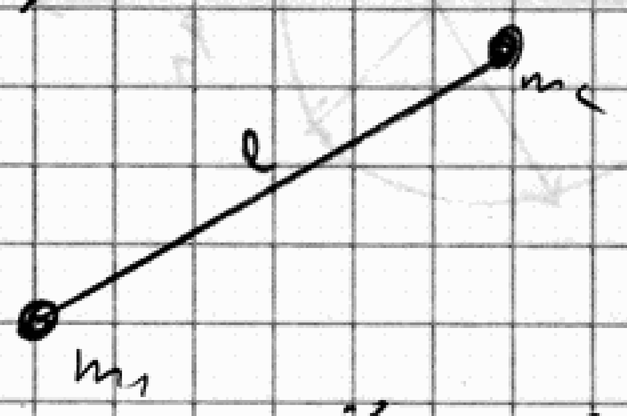
$$= P_1 - \frac{m_1}{M}\vec{P}_1 = \frac{m_2}{M}\vec{P}_1$$



$$Q_{max} = \frac{\vec{p}_1^{\,2}}{2\mu} = \frac{\mu}{2}(\vec{v}_1 - \vec{v}_2)^2$$

3)

a)



+ Schwerefeld d. Erde

$$\vec r_s = \frac{m_1 \vec r_1 + m_2 \vec r_2}{m_1 + m_2}$$

$m\vec g = m\ddot{\vec r}_s$ $\qquad$ $\vec v(t_0) = 0 \to c_1 = 0$

$mgt + c_0 = m\dot{\vec r}_s$ $\qquad$ $\vec v(t_0) = \vec v_0$ $\qquad$ $c_0 = m\vec v_0$

$\frac{1}{2} mgt^2 + c_0 t + c_1 = m\vec r_s$

$$\vec r_s = \frac{1}{2} g t^2 + \vec v_0 t$$

b) $\vec L = \vec r \times \vec p$

$\vec L = (\vec r_1 \times \dot{\vec r}_1) m_1 + (\vec r_2 \times \dot{\vec r}_2) m_2 = m_1 \left(r_s + \frac{m_2}{M} \vec r\right) \times \left(\dot{\vec r}_s + \frac{m_2}{M} \dot{\vec r}\right) + m_2 (\vec r_2 \times \dot{\vec r}_2)$

$= m_1 (\vec r_s \times \dot{\vec r}_s) + \frac{m_1 m_2}{m_1 + m_2} (\vec r \times \dot{\vec r}) +$

$+ \frac{m_1 m_2}{m_1 + m_2} \left[(\vec r_s \times \dot{\vec r}) + (\vec r \times \dot{\vec r}_s) \right]$

$+ m_2 (\vec r \times \dot{\vec r}_s)$

$+ \dots$

$\vec r = \vec r_1 - \vec r_2$

$\vec r_1 = \vec r_s + \frac{m_2}{M} \vec r$

$\vec r_2 = \vec r_s - \frac{m_1}{M} \vec r$

$= (m_1 + m_2)(\vec r_s \times \dot{\vec r}_s) + \frac{m_1 m_2}{m_1 + m_2} (\vec r \times \dot{\vec r})$

$= \cancel{M} \vec L_s + \vec L_R$

$\vec L_s = M \left(\frac{1}{2} g t^2 + \vec v_0 t \right) \times (\vec g t + v_0)$

$= \frac{1}{2} M (\vec v_0 \times \vec g t^2)$

c) $\vec{r} = \vec{r}_1 - \vec{r}_2$

$$\ddot{\vec{r}} = g + F_{12}\frac{1}{m_1} - g - \vec{F}_{21}'\frac{1}{m_2}$$

$$\ddot{\vec{r}} = \frac{\vec{F}_{12}}{m_1} - \frac{\vec{F}_{21}}{m_2} = F_{12}\left(\frac{1}{m_1} + \frac{1}{m_2}\right)$$

$$\vec{F} \sim \vec{r} \quad \rightarrow \quad L_z = \text{const}$$

d) $\vec{r} = l\,\vec{e}_r \qquad \dot{\vec{r}} = l\,\dot{\vec{e}}_r = l\,\dot{\varphi}\,\vec{e}_\varphi$

$$\ddot{\vec{r}} = l\ddot{\varphi}\,\vec{e}_\varphi - l\dot{\varphi}^2\,\vec{e}_r$$

$$\ddot{\varphi} = 0 \qquad \dot{\varphi} = \omega$$

$$\ddot{\vec{r}} = -\omega^2 l\,\vec{e}_r = -\omega^2\vec{r}$$

$$\vec{r} = l\cos(\omega t)\,\vec{e}_x + l\sin(\omega t)\,\vec{e}_y$$

$$\Delta\vec{r}_1 = \vec{r}_1 - \vec{r}_s = \frac{m_2}{m_1+m_2}\vec{r} = \frac{m_2}{M}l\left[\cos(\omega t)\,\vec{e}_x + \sin(\omega t)\,\vec{e}_y\right]$$

$$\Delta\vec{r}_2 = \vec{r}_2 - \vec{r}_s = -\frac{m_1}{M}r = -\frac{m_1}{M}l\left[\cos(\omega t)\,\vec{e}_x + \sin(\omega t)\,\vec{e}_y\right]$$