6. Übung

4) a)

$$W_c = \int \vec{F}\, d\vec{r}$$

$$F = a(3x-y)\,\vec{e}_x + b(2y-x)\,\vec{e}_y$$

$$C:\ \vec{r} = 2c\cos\varphi\,\vec{e}_x + 3c\sin\varphi\,\vec{e}_y$$

$$d\vec{r} = -2c\sin\varphi\,\vec{e}_x + 3c\cos\varphi\,\vec{e}_y$$

$$W_c = \int_0^{2\pi} [a(6c\cos\varphi - 3c\sin\varphi)\vec{e}_x + b(6c\sin\varphi - 2c\cos\varphi)\vec{e}_y]$$
$$(-2c\sin\varphi\,\vec{e}_x + 3c\cos\varphi\,\vec{e}_y)\ d\varphi$$

$$= \int_0^{2\pi} [-12ac^2\cos\varphi\sin\varphi + 6ac^2\sin^2\varphi + 18bc^2\sin\varphi\cos\varphi - 6bc^2\cos^2\varphi]\, d\varphi$$

$$= \int_0^{2\pi} [\underbrace{\cos\varphi\sin\varphi}_{0}(18bc^2 - 12ac^2) + 6ac^2\underbrace{\sin^2\varphi}_{\pi} - 6bc^2\underbrace{\cos^2\varphi}_{\pi}]\, d\varphi$$

$$= 6\pi c^2(a-b)$$

b) Wenn $\operatorname{rot}\vec{F} = 0 \Rightarrow \exists\, U$

$$\begin{vmatrix} e_x & e_y & e_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ a(3x-y) & b(2y-x) & 0 \end{vmatrix} = 0 + \vec{e}_y\left(\frac{\partial}{\partial z} a(3x-y)\right) + \vec{e}_z \frac{\partial}{\partial x} b(2y-x)$$
$$-\vec{e}_x\left(\frac{\partial}{\partial z} b(2y-x)\right) - e_y\left(\frac{\partial}{\partial x} 0\right) - \vec{e}_z \frac{\partial}{\partial y} a(3x-y)$$

$$= \vec{e}_z\,(a-b) \Rightarrow a = b$$

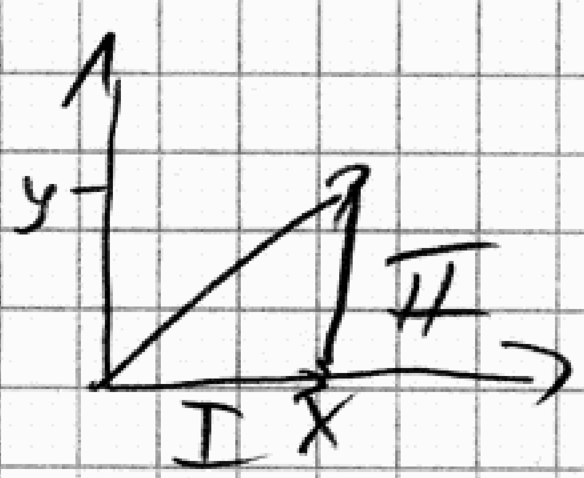
$$U = \int \vec{F}\cdot d\vec{r}$$

$$= \int_0^x F_x\, dx' + \int_0^y F_y\, dy'$$

$$= \int_0^x a(3x' - \overset{0}{y})\, dx' + \int_0^y b(2y' - x)\, dy'$$

$$= a\left(\frac{3x^2}{2} - xy\right) + a\left(\frac{2y^2}{2} - xy\right)$$

$$= a\,\frac{3x^2}{2} - 2xy + \frac{2y^2}{2}$$



d I $y = 0$

$$\vec{F} = -\operatorname{grad} U = -\vec{e}_x \frac{d}{dx}\, a\left(\frac{3x^2}{2} - 2xy + \frac{2y^2}{2}\right)$$
$$-\vec{e}_y \frac{d}{dy}\, a\left(\frac{3x^2}{2} - 2xy + \frac{2y^2}{2}\right)$$

$$= a(3x-y)\vec{e}_x + b(2y-x)$$

$$\vec{\nabla} \cdot r = \text{grad} \sqrt{\vec{r}\,\vec{r}} = \frac{\partial \sqrt{\vec{r}^2}}{d\vec{r}} = \frac{1}{2} \cdot 2 \sqrt{\vec{r}\,\vec{r}}$$

$$= \frac{\partial}{\partial x} \sqrt{x^2+y^2+z^2} + \frac{\partial}{\partial y} \sqrt{x^2+y^2+z^2} + \frac{\partial}{\partial z} (\sqrt{x^2+y^2+z^2})$$

$$= \frac{1}{2} \sqrt{x^2+y^2+z^2}^{\,-1} \cdot 2x$$

$$= \frac{x\vec{e}_x}{\sqrt{x^2+y^2+z^2}} + \frac{y\,\vec{e}_y}{\sqrt{\ldots}} + \frac{z\vec{e}_z}{\sqrt{\ldots}}$$

$$= \frac{\vec{r}}{|\vec{r}|} = \vec{e}_r$$

$$\vec{\nabla}(\vec{a} \cdot \vec{r}) = \vec{\nabla}(a_x x + a_y y + a_z z)$$

$$= a_x \vec{e}_x + a_y \vec{e}_y + a_z \vec{e}_z$$

1) $\vec{\nabla} r = \frac{\partial}{\partial \vec{r}} \sqrt{\vec{r}^2} = \frac{1}{2} \frac{1}{\sqrt{\vec{r}^2}} \cdot 2\vec{r} = \frac{\vec{r}}{\sqrt{\vec{r}^2}} = \vec{e}_r$

$$\vec{\nabla}(\vec{a} \cdot \vec{r}) = \cancel{\vec{a} \cdot \frac{\partial}{\partial \vec{r}} \circ \vec{r}}\ (\vec{\nabla} \cdot \vec{r})\ \vec{a} = 1\ \vec{a} = \vec{a}$$

$$\vec{\nabla} \frac{1}{r^2} = \frac{\partial}{\partial \vec{r}} \frac{1}{r^2} = \frac{\partial}{\partial r} \frac{1}{r^2} \cdot \frac{\partial r}{\partial \vec{r}} = -2 \cdot \frac{1}{r^3} \cdot \vec{e}_r$$

$$\vec{\nabla} f(r) = \frac{\partial}{\partial r} f(r) \cdot \frac{dr}{d\vec{r}} = f'(r)\ \vec{e}_r$$

$$\vec{\nabla} \cos(\vec{a} \cdot \vec{r}) = \frac{\partial}{\partial \vec{r}} \cos(\vec{a} \cdot \vec{r}) = -\sin(\vec{a} \cdot \vec{r}) \cdot \vec{a}$$

2)

$$\vec{\nabla} f(\vec{r} - \vec{r}') = \vec{\nabla}' f(\vec{r} - \vec{r}')$$

$$f'(\vec{r} - \vec{r}') \cdot 1 = -f'(\vec{r} - \vec{r}')(-1) \quad \checkmark$$

3) m_1, m_2

r_1, r_2

$$r = (\vec{r}_2 - \vec{r}_1)$$

$$U(r_2 - r_1) = -\gamma \frac{m_1 m_2}{|\vec{r}_2 - \vec{r}_1|}$$

$$\vec{F}_{12} = -\vec{\nabla} U(r_2 - \vec{r}_1)$$

$$= +\vec{\nabla}_r U(\vec{r})$$

$$= +\frac{dU(r)}{dr} \cdot \vec{\nabla}_r r$$

$$= +\gamma \frac{m_1 m_2}{|\vec{r}|^3} \vec{r}$$

$$= -\vec{F}_{21}$$

$$m_1 \ddot{\vec{r}}_1 = \vec{F}_{12}$$

$$m_2 \ddot{\vec{r}}_2 = \vec{F}_{21}$$

$$\vec{r}_s = \frac{m_1 \vec{r}_1 + m_2 \vec{r}_2}{m_1 + m_2}$$

$$\ddot{\vec{r}}_s = \frac{m_1 \ddot{\vec{r}}_1 + m_2 \ddot{\vec{r}}_2}{m_1 + m_2}$$

$$\ddot{\vec{r}}_s = \frac{\vec{F}_{12} + \vec{F}_{21}}{m_1 + m_2} = 0$$

$$\vec{r}_s(t) = \vec{v}_0 t + \vec{r}_0$$

<u>Relativ Bewegung:</u>

$$\vec{r}_1 = \vec{r}_s - \frac{m_2}{m_1 + m_2} \vec{r}$$

$$\vec{r}_2 = \vec{r}_s + \frac{m_1}{m_1 + m_2}$$

$$\ddot{\vec{r}} = \ddot{\vec{r}}_2 - \ddot{\vec{r}}_1 = \frac{\vec{F}_{21}}{m_2} - \frac{\vec{F}_{12}}{m_1}$$

$$= \frac{m_1 \vec{F}_{21} + m_2 \vec{F}_{12}}{m_1 m_2}$$

$$\ddot{\vec{r}} = \frac{1}{\mu} \vec{F}_{21}$$

$$\mu \ddot{\vec{r}} = \gamma \mu M \frac{\vec{r}}{r^3}$$

$$\rightarrow L = \vec{r} \times \mu \dot{\vec{r}} = \text{const}$$

$$= \mu (r \vec{e}_\rho \times (\dot{r} \vec{e}_\rho + r \underbrace{\dot{\vec{e}}_\rho}_{\dot{\varphi} \vec{e}_\varphi})$$

$$\vec{L}_r = \mu r^2 \dot{\varphi} \, \vec{e}_z$$

$$E_r = \frac{\mu}{2} \dot{\vec{r}}^2 + U(\vec{r}) = \frac{\mu}{2} \dot{r}^2 + \underbrace{\frac{\mu}{2} r^2 \dot{\varphi}^2}_{\frac{1}{2} \frac{L r^2}{\mu r^2}} + U(\varphi)$$

$$U_{eff} = \frac{1}{2} \frac{L^2}{\mu r^2} + U(r)$$

$$E_r = \frac{\mu}{2} \dot{r}^2 + U_{eff}(r)$$